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# On the Positivity of Correlations in Nonequilibrium Spin Systems

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We consider Ising spin systems, equivalently lattice gases evolving under discrete- or continuous-time Markov processes, i.e., "stochastic cellular automata" or "interacting particle systems." We show that for certain spin-flip probabilities or rates and suitable initial states the expectation values of products of spin variables taken at equal or different times are nonnegative; they satisfy the same inequalities as the equal-time correlations of ferromagnetic systems in equilibrium (first Griffiths inequality). Extensions of FKG inequalities to time-displaced correlations are also discussed.

**KEY WORDS:** Probabilistic cellular automata; interacting spin systems; correlation inequalities.

## 1. INTRODUCTION

Spin systems evolving in time under the action of stochastic dynamics are used to model phenomena as diverse as the structure of alloys and the functioning of neural networks.[1] While in some cases one is interested primarily in particular explicitly specified measures, whose stationarity under the time evolution is incidental, there are other cases where the only available information are the dynamical rules. Prime examples of the former are Gibbs measures, of the form $\exp[-\beta U(\sigma)]$, for some "local" interaction potential $U(\sigma)$. Here $\sigma = \{\sigma_x\}$, $\sigma_x = \pm 1$, $x \in Z^d$, and $\beta$ is the reciprocal temperature. Examples of the latter include various types of

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stochastic cellular automata used as models for computations with errors.[2]

In this paper we investigate extensions of what Gray[3] calls "statistical mechanical behavior" of Gibbs measures to more general processes. An example of such statistical mechanical behavior which we shall extend here to nonequilibrium systems is that of positivity of spin correlations in ferromagnetic systems. An equilibrium system is said to be *ferromagnetic* whenever all the interactions $J_R$ entering the potential are positive, i.e., if the finite-volume measures $\mu_\Lambda$ are of the form

$$\mu_\Lambda(\sigma)=\frac{1}{Z_\Lambda}\exp\Bigl(\sum_{R\subset\Lambda} J_R\sigma_R\Bigr), \qquad J_R\geqslant 0 \tag{1.1}$$

Here $\sigma_R=\prod_R \sigma_x$ for $R$ a subset of a region $\Lambda$ in the $d$-dimensional lattice $Z^d$. For ferromagnetic systems the expectation values of the spin variables $\sigma_A=\prod_{x\in A}\sigma_x$ with respect to the Gibbs measure (1.1) satisfy the (first) Griffiths inequality

$$\langle\sigma_A\rangle_{\text{eq}}\geqslant 0 \tag{1.2}$$

and the Griffiths, Kelly, Sherman (GKS) inequalities

$$\langle\sigma_A\sigma_B\rangle_{\text{eq}}-\langle\sigma_A\rangle_{\text{eq}}\langle\sigma_B\rangle_{\text{eq}}=\frac{\partial}{\partial J_B}\langle\sigma_A\rangle_{\text{eq}}\geqslant 0 \tag{1.3}$$

for all finite regions $A$, $B$ in $\Lambda$. The inequalities persist in the infinite-volume limit $\Lambda\uparrow Z^d$.

These correlation inequalities (whose proof is almost by inspection[4]) form an important and elegant tool for the study of ferromagnetic spin systems. In particular, they can be used to prove important results about phase transitions in equilibrium systems.[4,5] To the best of our knowledge, there has been no discussion in the literature about extensions of the inequalities (1.2) and (1.3) to non-Gibbsian measures. This is somewhat surprising, given their usefulness for equilibrium systems and the fact that there is a natural extension for the somewhat related FKG (Fortuin, Kasteleyn, and Ginibre[6])-type inequalities to time evolutions with "attractive" dynamics. In fact, the validity of the FKG inequalities for Gibbs states with appropriate interactions (Section 5) follows directly from the time-dependent results by choosing suitable spin-flip rates, for which the Gibbs measure is stationary. It would certainly be nice to have similar results for the ferromagnetic inequalities (1.2) and (1.3).

In this paper we prove the Griffiths inequalities (1.2) for measures obtained from certain types of stochastic time evolutions of spin systems on a lattice.[7] Unfortunately, the class of processes for which our result holds turns out to be rather restrictive. In particular, they do not include (for $d > 1$) any of the known spin-flip dynamics leading to ferromagnetic Gibbs states. We therefore do not even know whether the *time-displaced* equilibrium correlations in a ferromagnetic system evolving according to these spin-flip dynamics are nonnegative, although we expect them to be so. In fact, the dynamics for which our results hold have a trivial "phase diagram": they are (exponentially) ergodic for the allowed region of parameters unless there is a trapping state. As a consequence, the domination arguments naturally associated with relations like (1.2) only go in one direction. They show uniqueness of the invariant state for certain time evolutions. In particular, they give upper bounds for the critical temperature of some equilibrium systems.

We also obtain, using the transcription from $d$-dimensional stochastic cellular automata to $(d+1)$-dimensional Gibbs states (see, for example, ref. 8) positive correlations (1.2) for systems whose interactions are not entirely ferromagnetic, i.e., some of the $J_R$ are negative. Finally, we discuss some extensions of the FKG inequalities to unequal-time correlation functions, e.g., $\langle \sigma_x^0 \sigma_y^t \rangle \geq \langle \sigma_x^0 \rangle \langle \sigma_y^t \rangle$, in suitable systems.

## 2. FORMULATION OF THE PROBLEM

Consider a spin system on the lattice $Z^d$ and let $\sigma^t = \{\sigma_x^t\} \in \{-1, +1\}^{Z^d} = \Omega$ be a spin configuration at time $t$. The configuration $\sigma$ evolves in time, discrete or continuous, according to certain stochastic dynamics, which are described below.

### 2.1. Discrete Time

We define a Markov process on $\Omega$, $\{\sigma^k\}_{k \geq 0}$ such that, given the configuration $\sigma^n$ at time $n$, the value of the spin at site $x$ is $+1$ at time $n+1$ with probability $p_x(\sigma^n)$, independent of other sites and the past history of the process. The probability $p_x(\sigma)$ can be written conveniently as

$$p_x(\sigma) = \tfrac{1}{2}[1 + h(x, \sigma)] \tag{2.1}$$

with $h(x, \sigma)$ a local function for each $x \in Z^d$ with absolute value $|h(x, \sigma)| \leq 1$. This is equivalent to saying that the spin at $x$ changes sign in the $n$th updating ($\sigma_x^{n+1} = -\sigma_x^n$) with a *spin-flip probability*

$$c(x, \sigma) = \tfrac{1}{2}[1 - \sigma_x h(x, \sigma)] \tag{2.2}$$

Note that the spin updating at each time step occurs independently at the different sites $x \in Z^d$. This updating will be stochastic unless $h(x, \sigma) = \pm 1$, i.e., we are describing the evolution of a stochastic (or probabilistic) cellular automaton on $Z^d$.

Starting from a measure $\mu = \mu_0$ at time zero, the measure at time $n+1$, $n = 0, 1, 2,...,$ is given recursively by the relation

$$\int f(\sigma)\, \mu_{n+1}(d\sigma) = \int \mathrm{Tr} \left\{ f(\sigma) \prod_x [1 + \sigma_x h(x, \eta)] \right\} \mu_n(d\eta) \qquad (2.3)$$

for any local function $f$ of the configuration. $\mathrm{Tr}[\cdot]$ denotes the (normalized) trace operation:

$$\mathrm{Tr}[\phi(\sigma)] = \prod_{x \in A} \left( \frac{1}{2} \sum_{\sigma_x = \pm 1} \right) \phi(\sigma)$$

where $A$ is the support of the function $\phi$. We also use the notation, $\forall n \in N$,

$$\langle f \rangle_{n,\mu} = \int f(\sigma)\, \mu_n(d\sigma) \qquad (2.4)$$

for the expectation value with respect to the measures defined in (2.3). If $\mu = \delta_\eta$ where $\eta \in \Omega$ is a configuration, we write (2.4) as $\langle f \rangle_{n,\eta}$. A measure $\nu$ is said to be stationary (or invariant) if $\nu_n = \nu_{n+1}$ in the sense of Eq. (2.3). The process is called *ergodic* if there is a measure $\nu$ such that $\mu_n \to^W \nu$ as $n \to \infty$ for all initial $\mu$. This measure $\nu$ is necessarily invariant.

## 2.2. Continuous Time

The dynamics is now specified by the *spin-flip rates* $c(x, \sigma)$; $c(x, \sigma)\, dt$ is the probability for the spin at site $x$ to flip in the time interval $(t, t+dt)$ given that the configuration at time $t$ is $\sigma$.[7] We can write $c(x, \sigma)$ in the form

$$c(x, \sigma) = \tfrac{1}{2} c(x)[1 - \sigma_x h(x, \sigma)] \qquad (2.5)$$

where $c(x) \geqslant 0$, $\sup c(x) < \infty$, and the functions $h(x, \cdot)$ are as in (2.1). As for the discrete-time processes, we write $\langle f \rangle_{t,\mu}$ for the expectation of a function $f$ with respect to the measure $\mu_t$ at time $t$ if the system is started initially with the measure $\mu_0 = \mu$.[7] The corresponding evolution equation is

$$\frac{d}{dt} \langle f \rangle_{t,\mu} = \sum_x \langle c(x, \sigma)[f(\sigma^x) - f(\sigma)] \rangle_{t,\mu} \qquad (2.6)$$

where $\sigma^x$ is the configuration obtained from $\sigma$ by flipping the spin at $x$.

## 3. POSITIVE-TYPE DYNAMICS

The dynamics defined in (2.2) or (2.5) is determined by the functions $h(x, \sigma)$, which can be written in the form

$$h(x, \sigma) = \sum_A p_A(x)\, \sigma_{A+x} \tag{3.1}$$

where $A + x$ is the set $A \subset Z^d$ translated by the vector $x$, and $\sum_A$ runs over certain specified regions in the neighborhood of the origin (possibly containing it). If the coefficients $p_A(x)$ are all positive, we say that the dynamics is of *positive type*. Translation-invariant rates correspond to $p_A(x)$ independent of $x$.

**Proposition 1.** A dynamics of positive type conserves positive correlations. More explicitly, suppose that the initial measure $\mu$ has all correlations positive, i.e., it satisfies the condition (1.2). Then, for a dynamics of positive type,

(i) $\langle \sigma_{A_1}^{t_1} \cdots \sigma_{A_m}^{t_m} \rangle_{t,\mu} \geqslant 0$

(ii) $\langle \sigma_{A_1}^{t_1} \cdots \sigma_{A_m}^{t_m} \rangle_{t,\mu}$

is a nondecreasing function of the parameters $\{p_B(x)\}$ defined in (3.1) and of the initial time correlations

for all $A_1, ..., A_m \subset Z^d$ and all times $t_1, ..., t_m \geqslant 0$

We present a simple and direct proof for both discrete- and continuous-time processes. For simplicity we assume that the process is translation invariant.

This proposition can also be derived from the duality theorem of Holley and Stroock[9] or from Theorem III.4.13 in ref. 7.

### 3.1. Discrete Time

It is easy to check that

$$\langle \sigma_A \rangle_{n,\mu} = \sum_B M_{AB} \langle \sigma_B \rangle_{n-1,\mu} = \sum_B (M^n)_{AB} \int d\mu\, \sigma_B \tag{3.2}$$

where $M_{AB} = \mathrm{Tr}[\prod_{x \in A} h(x, \sigma)\, \sigma_B]$ and $M^n$ is the $n$th power of the matrix $M$. Since, for positive dynamics, $M_{AB} \geqslant 0$ for all $A$ and $B$ and since $M_{AB}$ is a nondecreasing function of the parameters, the conclusion follows. The same argument shows that $\langle \prod_{m \geqslant 0} \sigma_{A_m}^m \rangle$ is a nonnegative, nondecreasing function of the $\{p_C\}$, i.e., spins at different times are also positively correlated and the correlations are monotone in the coupling coefficients.

### 3.2. Continuous Time

Let us set the time scale such that $c(x) = c = 1$ in (2.5). From (2.6) we have that

$$\frac{d}{dt}\langle \sigma_A \rangle_{t,\mu} = -|A|\ \langle \sigma_A \rangle_{t,\mu} + \sum_{x \in A} \langle h(x, \sigma)\, \sigma_x \sigma_A \rangle_{t,\mu}$$

and after substituting $h(x, \sigma)$ by (3.1), we get

$$\frac{d}{dt}\langle \sigma_A \rangle_{t,\mu} = -|A|\ \langle \sigma_A \rangle_{t,\mu} + \sum_{x \in A} \sum_{B} p_B \langle \sigma_{B+x} \sigma_x \sigma_A \rangle_{t,\mu} \tag{3.3}$$

Hence,

$$\langle \sigma_A \rangle_{t,\mu} = \int \mu(d\sigma)\, \sigma_A e^{-|A|t} + \sum_{x \in A} \sum_{B} p_B \int_0^t \langle \sigma_{B+x} \sigma_x \sigma_A \rangle_{s,\mu}\, e^{|A|(s-t)}\, ds$$

This implies that once the correlations are positive and monotone in the $\{p_X\}$, they remain so at all later times. A more explicit solution to (3.3) can be obtained using duality and can be found in refs. 7 and 9. The above argument can be repeated for the case of unequal-time correlations.

## 4. APPLICATIONS, GENERALIZATIONS, AND CAUTIONARY EXAMPLES

### 4.1. Applications

**1.** Notice that the inequality $\sum_A |p_A(x)| = \sum_A p_A(x) \leqslant 1$ must hold for a dynamics of positive type [otherwise $c(x, \sigma)$ could be negative]. This implies that these processes have a unique invariant state whenever there is no trapping state, i.e., when the inequality is strict. This is a direct consequence of the following result:

**Proposition 2** (ref. 9; Theorem III.5.1 in ref. 7). With the definitions (3.1) and (2.5), suppose

$$\inf_x c(x) \left[ 1 - \sum_A |p_A(x)| \right] = \delta > 0$$

Then the process as defined in (2.6) is exponentially ergodic, i.e., for any probability measure $\mu$ on $\Omega$,

$$\left| \langle \sigma_A \rangle_{t,\mu} - \int d\nu\, \sigma_A \right| \leqslant 2e^{-\delta t}$$

where $\nu$ is the unique invariant measure.

The proof of Proposition 2 can be found in refs. 7 and 9. The proposition can be used to give upper bounds on the critical temperature of some equilibrium systems. The bound one obtains depends on the specific choice of spin-flip rate (and many different dynamics give rise to the same invariant measures). We illustrate this now by giving three examples of dynamics which have the Gibbs states of the two-dimensional nearest neighbor ferromagnetic Ising model at inverse temperature $\beta$ as their invariant measures. The critical temperature is exactly known: $\beta_c = 0.44...$ .

a. The standard Glauber rates[12]:

$$\begin{aligned} c(0, \sigma) &= \tfrac{1}{4}(1 - \sigma_0)[1 + \tfrac{1}{2}\gamma(\sigma_1 + \sigma_3)][1 + \tfrac{1}{2}\gamma(\sigma_2 + \sigma_4)] \\ &\quad + \tfrac{1}{4}(1 + \sigma_0)[1 - \tfrac{1}{2}\gamma(\sigma_1 + \sigma_3)][1 - \tfrac{1}{2}\gamma(\sigma_2 + \sigma_4)] \\ &= \tfrac{1}{2}[1 + \tfrac{1}{4}\gamma^2(\sigma_1 + \sigma_3)(\sigma_2 + \sigma_4) \\ &\quad - \tfrac{1}{2}\gamma\sigma_0(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)], \qquad \gamma = \tanh 2\beta \end{aligned}$$

where $\sigma_1$, $\sigma_3$ ($\sigma_2, \sigma_4$) are the left and right (upper and lower) nearest neighbor spins of the spin $\sigma_0$. The bound on the critical temperature obtained via Proposition 2 by requiring $\gamma^2 + 2\gamma < 1$ for ergodicity of the dynamics is $\gamma_c \geqslant \sqrt{2} - 1$, or $\beta_c \geqslant 0.19$.

b. Analogously (and using the same notation as above), we can obtain the same bound $\beta_c \geqslant 0.19$ for the rates considered by Künsch[13]:

$$c(0, \sigma) = \exp[-2\beta\sigma_0(\sigma_3 + \sigma_2)] = \cosh^2 2\beta[1 - \gamma\sigma_0(\sigma_3 + \sigma_2) + \gamma^2\sigma_2\sigma_3]$$

c. On the other hand, the stochastic Ising model corresponding to the spin-flip rates

$$\begin{aligned} c(0, \sigma) &= \{1 + \exp[2\beta\sigma_0(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)]\}^{-1} \\ &= \tfrac{1}{2}[1 - \tfrac{1}{8}(\tanh 4\beta + 2 \tanh 2\beta)\, \sigma_0(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4) \\ &\quad - \tfrac{1}{8}(\tanh 4\beta - 2 \tanh 2\beta)\, \sigma_0(\sigma_1\sigma_2\sigma_3 + \sigma_1\sigma_2\sigma_4 + \sigma_1\sigma_3\sigma_4 + \sigma_2\sigma_3\sigma_4)] \end{aligned}$$

gives a better bound on the critical temperature: $\tanh 2\beta_c \geqslant 0.5$ or $\beta_c \geqslant 0.27$.

**2.** It is possible to make a connection between a discrete-time process and an equilibrium system in one dimension higher by writing the transition function as an exponential of a local interaction.[8] For example, a one-dimensional voter model with independent spin flips has $h(x, \sigma) = \frac{1}{2}\alpha(\sigma_{x-1} + \sigma_{x+1})$, $0 \leqslant \alpha \leqslant 1$. We rewrite from (2.1)

$$\begin{aligned} \operatorname{Prob}(\sigma_x^{n+1} \mid \sigma^n) &= \frac{1}{2}\left[1 + \frac{\alpha}{2}(\sigma_{x-1}^n + \sigma_{x+1}^n)\, \sigma_x^{n+1}\right] \\ &= \frac{1}{N}\exp[\beta\sigma_x^{n+1}(\sigma_{x-1}^n + \sigma_{x+1}^n) - \beta J\sigma_{x-1}^n\sigma_{x+1}^n] \end{aligned}$$

with

$$\beta = \frac{1}{4} \log \frac{1+\alpha}{1-\alpha} \geqslant 0$$

$$\beta J = -\frac{1}{4} \log(1-\alpha^2) \geqslant 0$$

and

$$N = 2/(1-\alpha^2)^{1/4}$$

In this way the probability of a space-time configuration of the dynamics is given by the Gibbs measure [as in (1.1)] with Hamiltonian

$$-\beta H(\sigma) = \sum_{x,n} [\beta \sigma_x^{n+1}(\sigma_{x-1}^n + \sigma_{x+1}^n) - \beta J \sigma_{x-1}^n \sigma_{x+1}^n]$$

Despite the fact that this two-dimensional spin system is not ferromagnetic ($J \geqslant 0$), the correlations are still positive by Proposition 1 (see ref. 10 for a similar observation).

## 4.2. Generalizations

It is clear that these arguments can be generalized to the non-translation-invariant case. We also note that:

1. The coefficients $\{p_A(x)\}$ may depend on time as well. In particular, the $h(x, \sigma)$ can be periodic or random variables. This enables us to consider a joint process $(\sigma^t, \xi^t)$, where $\xi^t$ is some independently specified random process, which for fixed $\xi^t$ trajectories gives rise to a positive-type process for $\sigma^t$.

2. The assertions of Propositions 1 and 2 remain valid if an exchange process (or a Kawasaki dynamics at infinite temperature in the physics language[11]) is added to the dynamics. The generator of such a process is given by

$$Lf(\sigma) = \frac{1}{2} \sum_{\langle xy \rangle} [f(\sigma^{x,y}) - f(\sigma)] \tag{4.1}$$

where

$$\begin{aligned} (\sigma^{x,y})_z &= \sigma_z && \text{if} \quad x \neq z \neq y \\ &= \sigma_y && \text{if} \quad z = x \\ &= \sigma_x && \text{if} \quad z = y \end{aligned}$$

is the configuration obtained by exchanging the spins at sites $x$ and $y$. The sum in (4.1) is over nearest neighbor pairs. There is no additional complication if this sum is made asymmetric, i.e., if there is a different probability to exchange the spin values for the different neighbors. Notice in particular that the dynamics in paragraphs a, b, and c remain ergodic when such an exchange process is added for the same range of temperature as was specified there.

## 4.3. Cautionary Examples

1. Having positive correlations in the initial state is in general not sufficient for dynamics which have ferromagnetic Gibbs states as their stationary measures. Consider a system of three spins $\sigma_1$, $\sigma_2$, and $\sigma_0$ with spin-flip rates $c(i, \sigma) = \exp[-\beta\sigma_i(\sigma_j + \sigma_k)]$ for $i$, $j$, and $k$ different elements of $\{0, 1, 2\}$. The stationary state is $\nu \sim \exp[\beta(\sigma_1\sigma_2 + \sigma_1\sigma_2 + \sigma_2\sigma_3)]$. We can explicitly write down the equations for the correlations at any time $t$. The result is that positive correlations are not conserved by the time evolution. A sufficient condition on the initial measure $\mu$, besides having positive correlations, to get positive correlations at all times is given by $\int d\mu \operatorname{sgn}(\sigma_0 + \sigma_1 + \sigma_2) \geqslant 0$. This condition is also a necessary condition at zero temperature ($\beta = +\infty$). We expect that starting from all spins up at time zero is sufficient to get positive correlations for the infinite system at all later times.

2. A dynamics of positive type does not necessarily conserve the "second Griffiths inequality" (1.3). The analog of (1.3) would say that a dynamics of positive type started from a measure satisfying (1.3) satisfies the inequalities

$$\langle \sigma_A \sigma_B \rangle_{t,\mu} - \langle \sigma_A \rangle_{t,\mu} \langle \sigma_B \rangle_{t,\mu} \geqslant 0 \tag{4.2}$$

This is, however, not true. We give a counterexample in one dimension. Let $h(x, \sigma) = \frac{1}{2}(\tanh \gamma)(\sigma_x + \sigma_{x+1})$. Then the process has as its unique stationary measure the Gibbs measure associated with a nearest neighbor pair potential $J = \frac{1}{2} \log \cosh 2\gamma$. This Gibbs measure obviously satisfies (1.3). However, if the initial $\mu$ is a Bernoulli measure with density 1/2, then, for all $x \in Z$ and $n \geqslant 1$, $i > n$, with $\lambda = \tanh \gamma$,

$$\langle \sigma_x \sigma_{x+i} \rangle_{n,\mu} = (\lambda/2)^2 \langle (\sigma_x + \sigma_{x+1})(\sigma_{x+i} + \sigma_{x+i+1}) \rangle_{n-1,\mu} = 0$$

but

$$\langle \sigma_x \sigma_{x+n} \rangle_{n,\mu} = (\lambda/2)^2 \langle (\sigma_x + \sigma_{x+1})(\sigma_{x+n} + \sigma_{x+n+1}) \rangle_{n-1,\mu} = (\lambda/2)^{2n}$$

so that (4.2) fails for $A = \{x, x+t\}$, $B = \{x+t, x+t+1\}$, and (discrete time) $t \geqslant 1$.

## 5. ATTRACTIVE DYNAMICS

The dynamics corresponding to the spin flip rates of paragraphs a, b, and c of the previous section are *attractive*, which means roughly that the dynamics likes to line up the spins parallel to each other. To be more precise, we say that a function $f(\sigma)$ is *nondecreasing* if $f(\sigma) \geqslant f(\eta)$ whenever $\sigma_x \geqslant \eta_x$ for all $x \in Z^d$. Let $S(t)\, f(\eta)$ be the expectation value of $f(\sigma^t)$ if the configuration at time zero was $\sigma^0 = \eta$, i.e., $S(t)\, f(\eta) = \langle f \rangle_{t,\eta}$. An *attractive* time evolution is then one which leaves the set of nondecreasing functions invariant: $S(t) f$ is nondecreasing for all times $t$ if $f$ is nondecreasing. An easily verifiable sufficient condition for the attractiveness of a dynamics [using the rates (2.5)] is that the functions $\{h(x, \sigma)\}$ are nondecreasing. An attractive dynamics has the property that it preserves positive correlations between increasing functions, i.e., if the initial measure $\mu$ satisfies the conditions of the FKG inequalities[6]

$$\int d\mu\, fg \geqslant \int d\mu\, f \int d\mu\, g \tag{5.1}$$

for any two nondecreasing functions $f$ and $g$, then this holds also at all later times: for all $t \geqslant 0$,

$$\langle fg \rangle_{t,\mu} - \langle f \rangle_{t,\mu} \langle g \rangle_{t,\mu} \geqslant 0 \tag{5.2}$$

Examples of such measures $\mu$ are the delta measures $\delta_+$, resp. $\delta_-$, concentrated on the configuration with all spin values $+1$, resp. all spin values $-1$, and the Bernoulli measure.

The inequalities (5.2) remain valid also in the limit $t \uparrow \infty$. In particular, if $v$ is an invariant measure obtained in this way from such a $\mu$, then it satisfies the FKG inequalities (5.1). This can be used to prove that the Gibbs measure

$$v_{\text{eq}} \sim \exp\left[\beta \sum J(x, y)\, \sigma_x \sigma_y + \beta \sum h(x)\, \sigma_x\right]$$

with $J(x, y) \geqslant 0$ satisfies the FKG inequalities (5.1). Clearly, if $v$ is translation invariant, then its time-displaced correlation functions are positive,

$$\langle \sigma_x^0 \sigma_y^t \rangle \geqslant \langle \sigma_x^0 \rangle \langle \sigma_y^t \rangle = m^2 \geqslant 0 \tag{5.3}$$

where for simplicity we write $\langle \cdot \rangle$ for $\int v(d\sigma) \cdot$ and $\langle \sigma_x \rangle = m$. We note that FKG inequalities can be used to derive bounds on the decay of the correlations between two widely separated sets (in space and time) of spins in terms of the decay of time-displaced pair correlations. This follows from arguments first described in ref. 14 for equilibrium systems. Let $T$ and $S$ be two space-time sets. Define $\rho_R = \prod_R \rho_x^t$, $\rho_x^t = \frac{1}{2}(1 + \sigma_x^t)$ for $R = T$ or $S$, where $\sigma_x^t$ is the spin at site $x$ at time $t$. Then it folows that

$$0 \leqslant \langle \rho_T \rho_S \rangle - \langle \rho_T \rangle \langle \rho_S \rangle \leqslant \tfrac{1}{4} |T|\, |S|\, u(T, S) \tag{5.4}$$

where $|T|$ and $|S|$ are the cardinalities of the sets $T$ and $S$ and the function $u$ is any upper bound on $\langle \sigma_x^s \sigma_y^t \rangle - \langle \sigma_x^s \rangle \langle \sigma_y^t \rangle$, $\forall (x, s) \in T$, $(y, t) \in S$.

The inequalities (5.4) can be used to obtain bounds on certain dynamical exponents. Thus, the energy–energy correlations $\langle \sigma_x^t \sigma_y^t \sigma_x^0 \sigma_y^0 \rangle - \langle \sigma_x^t \sigma_y^t \rangle \langle \sigma_x^0 \sigma_y^0 \rangle$ in a spin system with ferromagnetic pair interactions must decay in time at least as fast as the spin–spin correlations $\langle \sigma_z^t \sigma_u^0 \rangle - \langle \sigma_z^t \rangle \langle \sigma_u^0 \rangle$.

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